

Exact Controllability for Stochastic Schrödinger Equations*

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Abstract

This paper is addressed to studying the exact controllability for stochastic Schrödinger equations by two controls. One is a boundary control in the drift term and the other is an internal control in the diffusion term. By means of the standard duality argument, the control problem is converted into an observability problem for backward stochastic Schrödinger equations, and the desired observability estimate is obtained by a global Carleman estimate. At last, we give a result about the lack of exact controllability, which shows that the action of two controls is necessary.

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1 Introduction

Let $T > 0$, $G \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a given bounded domain with the C^2 boundary Γ . Let Γ_0 be a suitable chosen nonempty subset (to be given later) of Γ . Put $Q \triangleq (0, T) \times G$, $\Sigma \triangleq (0, T) \times \Gamma$ and $\Sigma_0 \triangleq (0, T) \times \Gamma_0$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined such that $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration generated by $\{B(t)\}_{t \geq 0}$, augmented by all the P -null sets in \mathcal{F} . Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0, T; H)}) < \infty$; by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted bounded processes; and by $C_{\mathcal{F}}([0, T]; L^2(\Omega; H))$ the Banach space

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consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes $X(\cdot)$ such that $|X(\cdot)|_{L^2(\Omega; H)} \in C([0, T])$. All of the above spaces are endowed with the canonical norm.

Denote by $\nu(x)$ the unit outward normal vector of G at $x \in \Gamma$. Let $x_0 \in (\mathbb{R}^n \setminus \overline{G})$. In what follows, we choose

$$\Gamma_0 = \{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}. \quad (1.1)$$

The main purpose of this paper is to study the exact controllability of the following controlled linear stochastic Schrödinger equation

$$\begin{cases} idy + \Delta y dt = (a_1 \cdot \nabla y + a_2 y + f)dt + (a_3 y + g)dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma \setminus \Sigma_0, \\ y = u & \text{on } \Sigma_0, \\ y(0) = y_0 & \text{in } G. \end{cases} \quad (1.2)$$

Here, the initial state $y_0 \in L^2(\Omega, \mathcal{F}_0, P; H^{-1}(G))$, the control

$$u \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)), \quad g \in L^2(0, T; H^{-1}(G)),$$

the nonhomogeneous term $f \in L^2_{\mathcal{F}}(0, T; L^2(G))$ and the coefficients $a_i (i = 1, 2, 3)$ satisfy

$$\begin{cases} ia_1 \in L^\infty_{\mathcal{F}}(0, T; W^{2,\infty}(G; \mathbb{R}^n) \cap W^{1,\infty}_0(G; \mathbb{R}^n)), \\ a_2 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G)), \\ a_3 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G)). \end{cases} \quad (1.3)$$

System (1.2) has a nonhomogeneous boundary condition. As the deterministic nonhomogeneous boundary problem, the solution to (1.2) is understood in the transposition sense. Hence, we first introduce the following backward stochastic Schrödinger equation

$$\begin{cases} idz + \Delta z dt = (b_1 \cdot \nabla z + b_2 z + b_3 Z)dt + ZdB(t) & \text{in } (0, \tau) \times G, \\ z = 0 & \text{on } (0, \tau) \times \Gamma, \\ z(\tau) = z_\tau & \text{in } G, \end{cases} \quad (1.4)$$

where $z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))$, the coefficients $b_i (i = 1, 2, 3)$ satisfy

$$\begin{cases} ib_1 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}_0(G; \mathbb{R}^n)), \\ b_2 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G)), \\ b_3 \in L^\infty_{\mathcal{F}}(0, T; W^{1,\infty}(G)). \end{cases} \quad (1.5)$$

For the convenience of the reader, we recall the definition of the solution to (1.4) first.

Definition 1.1 *A solution to the equation (1.4) is a pair of stochastic processes*

$$(z, Z) \in L^\infty_{\mathcal{F}}(0, T; H_0^1(G)) \times L^2_{\mathcal{F}}(0, T; H_0^1(G))$$

such that for every $\psi \in C_0^\infty(G)$ and a.e. $(t, \omega) \in [0, \tau] \times \Omega$, it holds that

$$\begin{aligned} & \int_G i z_\tau(x) \psi(x) dx - \int_G i z(t) \psi(x) dx - \int_t^\tau \int_G \nabla z(s, x) \cdot \nabla \psi(x) dx ds \\ &= \int_t^\tau \int_G [b_1(s, x) \nabla z(s, x) + b_2(s, x) z(s, x) + b_3(s, x) Z(s, x)] \psi(x) dx ds \\ &+ \int_t^\tau \int_G Z(s, x) \psi(x) dx dB(s). \end{aligned} \quad (1.6)$$

Let us recall the well-posedness result of (1.4) (see [1, 19] for the proof).

Lemma 1.1 *For any $z_\tau \in L^2(\Omega, \mathcal{F}_\tau, P; H_0^1(G))$, the equation (1.4) admits a unique solution (z, Z) . Moreover, (z, Z) satisfies that*

$$|z|_{L_{\mathcal{F}}^\infty(0, \tau; H_0^1(G))} + |Z|_{L_{\mathcal{F}}^2(0, \tau; H_0^1(G))} \leq e^{Cr_1} |z_\tau|_{L^2(\Omega, \mathcal{F}_\tau, P; H_0^1(G))}, \quad (1.7)$$

where

$$r_1 \triangleq |b_1|_{L_{\mathcal{F}}^\infty(0, T; W^{1, \infty}(G; \mathbb{R}^n))}^2 + \sum_{i=2}^3 |b_i|_{L_{\mathcal{F}}^\infty(0, T; W^{1, \infty}(G))}^2 + 1.$$

Here and in the sequel, we will use C to denote a generic positive constant depends on T , G , Γ_0 (unless otherwise stated), which may vary from line to line.

Further, we need the following result for the hidden regularity of the solution to (1.4).

Proposition 1.1 *Let z be a solution to (1.4), then $\frac{\partial z}{\partial \nu}|_\Gamma \in L_{\mathcal{F}}^2(0, \tau; L^2(\Gamma))$. Further, we have the following estimate*

$$\left| \frac{\partial z}{\partial \nu} \right|_{L_{\mathcal{F}}^2(0, \tau; L^2(\Gamma))} \leq e^{Cr_1} |z_\tau|_{L_{\mathcal{F}}^2(\Omega; H_0^1(G))}. \quad (1.8)$$

Remark 1.1 *Proposition 1.1 shows that, solutions of (1.4) enjoy a higher regularity on the boundary than the one provided by the classical trace theorem of Sobolev spaces. Such kind of result is called hidden regularity of the solution. There are a great many studies in this topic for deterministic partial differential equations in the literature (see [12, 15] for example).*

Now we can give the definition of the solution to (1.2).

Definition 1.2 *A solution to the system (1.2) is a process $y \in C_{\mathcal{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ such that for every $\tau \in [0, T]$ and every $z_\tau \in L_{\mathcal{F}_\tau}^2(\Omega; H_0^1(G))$ it holds that*

$$\begin{aligned} & \mathbb{E} \int_G y(\tau, x) z_\tau(x) dx - \mathbb{E} \int_G y_0(x) z(0) dx \\ &= \mathbb{E} \int_0^\tau \int_{\Gamma_0} \frac{\partial z}{\partial \nu} u dx ds + \mathbb{E} \int_0^\tau \int_G f z dx dt + \mathbb{E} \int_0^\tau \int_G g Z dx dt. \end{aligned} \quad (1.9)$$

Here (z, Z) solves (1.4) with

$$b_1 = -a_1, \quad b_2 = -\operatorname{div}(a_1) + a_2, \quad b_3 = -a_3.$$

Remark 1.2 *The solution to (1.2) is defined in the transposition sense. It is well studied that such kind of solutions for deterministic nonhomogeneous boundary value problems in the literature (see [15, 16] for example). On the other hand, the stochastic counterpart is almost open. We only consider a very special problem in this topic. The study of general stochastic nonhomogeneous boundary value problems are very interesting but difficult problems, which is far beyond the scope of this paper.*

We have the following well-posedness result for (1.2).

Proposition 1.2 *For each $y_0 \in L^2_{\mathcal{F}_0}(\Omega; H^{-1}(G))$, the system (1.2) admits a unique solution y . Further, for every $y_0 \in L^2_{\mathcal{F}_0}(\Omega; H^{-1}(G))$, it holds that*

$$\begin{aligned} & |y|_{C_{\mathcal{F}}([0,T]; L^2(\Omega; H^{-1}(G)))} \\ & \leq e^{Cr_1} \left(\mathbb{E}|y_0|_{H^{-1}(G)} + |f|_{L^2_{\mathcal{F}}(0,T; L^2(G))} + |u|_{L^2_{\mathcal{F}}(0,T; L^2(\Gamma_0))} + |g|_{L^2_{\mathcal{F}}(0,T; H^{-1}(G))} \right). \end{aligned} \quad (1.10)$$

Here

$$r_2 = |a_1|_{L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}_0(G; \mathbb{R}^n))}^2 + |a_2|_{L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}(G))}^2 + |a_3|_{L^\infty_{\mathcal{F}}(0,T; W^{1,\infty}(G))}^2 + 1.$$

Now we can give the definition of the exact controllability of (1.2).

Definition 1.3 *System (1.2) is said to be exactly controllable at time T if for every initial state $y_0 \in L^2_{\mathcal{F}_0}(\Omega; H^{-1}(G))$ and every $y_1 \in L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$, one can find a pair of controls $(u, g) \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathcal{F}}(0, T; H^{-1}(G))$ such that the solution y of the system (1.2) satisfies that $y(T) = y_1$ in $L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$.*

As the deterministic case, the exact controllability of (1.2) can be reduced to an observability estimate of its dual system, that is, the equation (1.4) with $\tau = T$ in our case. For the latter one, we will prove the following result.

Theorem 1.1 *All solutions of the equation (1.4) satisfy that*

$$|z_T|_{L^2_{\mathcal{F}_T}(\Omega; H^1_0(G))} \leq e^{Cr_1} \left(|z|_{L^2_{\mathcal{F}}(0,T; L^2(\Gamma_0))} + |Z|_{L^2_{\mathcal{F}}(0,T; H^1_0(G))} \right). \quad (1.11)$$

By means of Theorem 1.1, we can obtain the following the exact controllability result of the system (1.2).

Theorem 1.2 *System (1.2) is exactly controllable at any time $T > 0$.*

Further, we also have the following result about the lack of exact controllability result for (1.2) if the control u in the drift term is zero.

Theorem 1.3 *If $u \equiv 0$ in (1.2), then (1.2) is not exactly controllable at any time T .*

Remark 1.3 *According to the lack of exact controllability for linear stochastic ordinary differential equations in [21], it is clear that the internal control g is necessary for the exact controllability of the system (1.2). From Theorem 1.3, we know that u is also necessary. Hence, we have that one should utilize both u and g to get the exact controllability of (1.2).*

There exist many approaches and results addressing the controllability problem for deterministic Schrödinger equations. [24] is a nice survey for the works in this respect before 2002. For the works after 2002, we refer the readers to [3, 4, 5, 6, 8, 22] and the rich references therein. However, people know very little about the stochastic counterpart. To our best knowledge, there is no published result for this problem.

Generally speaking, there are four main methods for the exact controllability of deterministic Schrödinger equations.

The first one is the classical Rellich-type multiplier approach ([20]). It can be applied to treat Schrödinger equations with no lower order terms or lower order terms with constant or small coefficients. On the other hand, it seems that it cannot be used to solve our problem since we do not assume that the coefficients of lower order terms are constant or small.

The second one is the microlocal analysis approach ([14]). This method was first introduced in [2] for obtaining the exact controllability of wave equations. It is useful to solve the exact controllability problem for many kinds of partial differential equations such as wave equations, Schrödinger equations and plate equations. However, it seems that there are lots of obstacles needing to be surmounted if one utilize it to study the stochastic control problem (see remarks in Section 6 for more details).

The third one is based on the Ingham type inequality ([11]). This method works well for Schrödinger equations involved in some special domains, i.e., intervals, rectangles and balls. However, it seems that it is very hard to applied to equations in general domains.

The last one is the global Carleman estimate ([13]). It can be regarded as a more developed version of the classical multiplier method. With respect to the method of multipliers, the Carleman approach has the advantage of being more flexible and allowing to address variable coefficients, with respect to the microlocal one, that it requires less regularity on coefficients and domain, and respect to the method based on Ingham type inequality, that is requires less restrictions to the domain. Further, it is robust with respect to the lower order terms and can be used to get explicit bounds on the observability constant/control cost in terms of the potentials entering in it. This is particularly important when dealing with nonlinear problems by combing linearization and fixed point techniques.

Similar to the deterministic setting, we use a stochastic version of the global Carleman estimate to derive Theorem 1.1. For this, we borrow some idea from the proof of the observability estimate for deterministic Schrödinger equations (see [13] for example). However, the stochastic setting will produce some more troubles. We cannot simply mimic the method in [13] to solve our problem. To handle these troubles, we choose a different weight function for the Carleman estimate.

2 Some preliminaries

In this section, we give some preliminary results.

First, we prove the well-posedness of (1.2) in the sense of Definition 1.2.

Proof of Proposition 1.2: Uniqueness of the solution. Suppose there are $y_1(\cdot)$ and $y_2(\cdot)$ belong to $C_{\mathcal{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$ such that (1.9) holds. Then, we see

$$\mathbb{E} \int_G y_1(\tau, x) z_\tau(x) dx = \mathbb{E} \int_G y_2(\tau, x) z_\tau(x) dx \quad \text{for all } z_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; H^1_0(G)).$$

This concludes that $y_1 = y_2$.

Existence of the solution. Let us define a linear functional F on $L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))$ as

$$F(z_\tau) = (y_0, z(0))_{H^{-1}(G), H_0^1(G)} + \mathbb{E} \int_0^\tau \int_{\Gamma_0} \frac{\partial z}{\partial \nu} u dx ds + \mathbb{E} \int_0^\tau \int_G f z dx dt + \mathbb{E} \int_0^\tau \int_G g Z dx dt.$$

It is easy to show that F is a bounded linear functional on $L^2_{\mathcal{F}_\tau}(\Omega; H_0^1(G))$. By Riesz Representation Theorem, we know there exists a $y_\tau \in L^2_{\mathcal{F}_\tau}(\Omega; H^{-1}(G))$ such that

$$F(z_\tau) = \mathbb{E}(y_\tau, z_\tau)_{H^{-1}(G), H_0^1(G)}.$$

Define a process $y(\cdot)$ by $y(\tau) = y_\tau$. Now we prove that $y(\cdot) \in C_{\mathcal{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$.

Let $\xi \in L^2_{\mathcal{F}_T}(\Omega; H_0^1(G))$. Consider the following forward random Schrödinger equation

$$\begin{cases} id\tilde{z} + \Delta\tilde{z}dt = (-a_1 \cdot \nabla\tilde{z} - \operatorname{div} a_1\tilde{z} + a_2\tilde{z})dt & \text{in } (\tau, \tau + \delta) \times G, \\ \tilde{z} = 0 & \text{on } (\tau, \tau + \delta) \times \Gamma, \\ \tilde{z}(\tau) = \mathbb{E}(\xi|\mathcal{F}_\tau) & \text{in } G, \end{cases} \quad (2.1)$$

It is easy to see that

$$\lim_{\delta \rightarrow 0^+} \mathbb{E}|\tilde{z}(\tau + \delta) - \tilde{z}(\tau)|_{H_0^1(G)}^2 = 0. \quad (2.2)$$

Further, since $\{\mathcal{F}_t\}_{t \geq 0}$ is the natural filtration of $\{B(t)\}_{t \geq 0}$, we have

$$\lim_{\delta \rightarrow 0^+} \mathbb{E}|\mathbb{E}(\xi|\mathcal{F}_{\tau+\delta}) - \mathbb{E}(\xi|\mathcal{F}_\tau)|_{H_0^1(G)}^2 = 0. \quad (2.3)$$

From (2.2) and (2.3), we see

$$\lim_{\delta \rightarrow 0^+} \mathbb{E}|\tilde{z}(\tau + \delta) - \mathbb{E}(z|\mathcal{F}_{\tau+\delta})|_{H_0^1(G)}^2 = 0. \quad (2.4)$$

Let $(z_1(\cdot), Z_1(\cdot))$ and $(z_2(\cdot), Z_2(\cdot))$ be the solution to (1.4) with final data $\mathbb{E}(\xi|\mathcal{F}_{\tau+\delta})$ and $\tilde{z}(\tau + \delta)$, respectively. Then, from Lemma 1.1 and Proposition 1.1, we know

$$\begin{cases} \lim_{\delta \rightarrow 0^+} |z_1 - z_2|_{L^2_{\mathcal{F}}(0, \tau; L^2(G))} = 0, \\ \lim_{\delta \rightarrow 0^+} |Z_1 - Z_2|_{L^2_{\mathcal{F}}(0, \tau; L^2(\Gamma_0))} = 0, \\ \lim_{\delta \rightarrow 0^+} \left| \frac{\partial z_1}{\partial \nu} - \frac{\partial z_2}{\partial \nu} \right|_{L^2_{\mathcal{F}}(0, \tau; L^2(\Gamma_0))} = 0, \\ \lim_{\delta \rightarrow 0^+} |z_1(0) - z_2(0)|_{H_0^1(G)} = 0. \end{cases} \quad (2.5)$$

Denote by (z_3, Z_3) the solution to (1.4) with the final datum $z_3(\tau) = \mathbb{E}(\xi|\mathcal{F}_\tau)$. From the uniqueness of the solution to (2.1) and (1.4), we find that

$$z_3 = z_2 \text{ in } [0, \tau] \times G \text{ and } Z_3 = Z_2 \text{ in } [0, \tau] \times G. \quad (2.6)$$

From the definition of the solution to (1.2), we have

$$\begin{aligned}
& \mathbb{E}(y(\tau + \delta) - y(\tau), \xi)_{H^{-1}(G), H_0^1(G)} \\
&= \mathbb{E}(y(\tau + \delta), \xi)_{H^{-1}(G), H_0^1(G)} - \mathbb{E}(y(\tau), \xi)_{H^{-1}(G), H_0^1(G)} \\
&= \mathbb{E}(y(\tau + \delta), \mathbb{E}(\xi | \mathcal{F}_{\tau+\delta}))_{H^{-1}(G), H_0^1(G)} - \mathbb{E}(y(\tau), \mathbb{E}(\xi | \mathcal{F}_\tau))_{H^{-1}(G), H_0^1(G)} \\
&= \mathbb{E}(y_0, z_1(0) - z_3(0))_{H^{-1}(G), H_0^1(G)} + \mathbb{E} \int_0^\tau \int_{\Gamma_0} \left(\frac{\partial z_1}{\partial \nu} - \frac{\partial z_3}{\partial \nu} \right) u d\Gamma ds \\
&\quad + \mathbb{E} \int_0^\tau \int_G f(z_1 - z_3) dx dt + \mathbb{E} \int_0^\tau \int_G g(Z_1 - Z_3) dx dt + \mathbb{E} \int_\tau^{\tau+\delta} \int_{\Gamma_0} \frac{\partial z_1}{\partial \nu} u d\Gamma ds \\
&\quad + \mathbb{E} \int_\tau^{\tau+\delta} \int_G f z_1 dx dt + \mathbb{E} \int_\tau^{\tau+\delta} \int_G g Z_1 dx dt.
\end{aligned}$$

This, together with (2.5) and (2.6), implies that

$$\lim_{\delta \rightarrow 0^+} \mathbb{E}(y(\tau + \delta) - y(\tau), \xi)_{H^{-1}(G), H_0^1(G)} = 0, \quad \text{for any } \xi \in L_{\mathcal{F}_T}^2(\Omega; H_0^1(G)).$$

Similarly, we can show that

$$\lim_{\delta \rightarrow 0^-} \mathbb{E}(y(\tau + \delta) - y(\tau), \xi)_{H^{-1}(G), H_0^1(G)} = 0, \quad \text{for any } \xi \in L_{\mathcal{F}_T}^2(\Omega; H_0^1(G)).$$

Hence, we see $y(\cdot) \in C_{\mathcal{F}}([0, T]; L^2(\Omega; H^{-1}(G)))$. \square

Next, for the sake of completeness, we give an energy estimate for the equation (1.4).

Proposition 2.1 *For all z which solve the equation (1.4), it holds that*

$$\mathbb{E}|z(t)|_{H_0^1(G)}^2 \leq e^{Cr_1} \left(\mathbb{E}|z(s)|_{H_0^1(G)}^2 + |Z|_{L_{\mathcal{F}}^2(0, T; H_0^1(G))}^2 \right), \quad (2.7)$$

for any $0 \leq s \leq t \leq \tau$.

Proof: By direct computation, we have

$$\begin{aligned}
& \mathbb{E}|z(t)|_{L^2(G)}^2 - \mathbb{E}|z(s)|_{L^2(G)}^2 \\
&= \mathbb{E} \int_s^t \int_G (z d\bar{z} + \bar{z} dz + dz d\bar{z}) dx \\
&= \mathbb{E} \int_s^t \int_G \left\{ -iz(\Delta \bar{z} + b_1 \cdot \nabla \bar{z} - b_2 \bar{z} - b_3 \bar{Z}) \right. \\
&\quad \left. + i\bar{z}(\Delta z - b_1 \cdot \nabla z - b_2 z - b_3 Z) + Z\bar{Z} \right\} dx d\sigma \\
&= \mathbb{E} \int_s^t \int_G \left\{ -i[\operatorname{div}(z \nabla \bar{z}) - |\nabla z|^2 + \operatorname{div}(|z|^2 b_1) - \operatorname{div}(b_1)|z|^2 - b_2|z|^2 - b_3 z \bar{Z}] \right. \\
&\quad \left. + i[\operatorname{div}(\bar{z} \nabla z) - |\nabla \bar{z}|^2 - b_2|z|^2 - b_3 \bar{z} Z] + |Z|^2 \right\} dx d\sigma \\
&\leq \mathbb{E} \int_s^t 2 \left[(|b_1|_{W^{1, \infty}(G; \mathbb{R}^n)} + |b_3|_{L^\infty(G)} + 1) |z|_{L^2(G)}^2 + |Z|_{L^2(G)}^2 \right] dx d\sigma
\end{aligned} \quad (2.8)$$

and

$$\begin{aligned}
& \mathbb{E}|\nabla z(t)|_{L^2(G)}^2 - \mathbb{E}|\nabla z(s)|_{L^2(G)}^2 \\
&= \mathbb{E} \int_s^t \int_G (\nabla z d\nabla \bar{z} + \nabla \bar{z} d\nabla z + d\nabla z d\nabla \bar{z}) dx \\
&= \mathbb{E} \int_s^t \int_G \left\{ \operatorname{div}(\nabla z d\bar{z}) - \Delta z d\bar{z} + \operatorname{div}(\nabla \bar{z} dz) - \Delta \bar{z} dz + d\nabla z d\nabla \bar{z} \right\} dx \\
&= \mathbb{E} \int_s^t \int_G \left\{ \Delta z \left[i(\Delta \bar{z} + b_1 \cdot \nabla \bar{z} - b_2 \bar{z} - b_3 \bar{Z}) \right] \right. \\
&\quad \left. - \Delta \bar{z} \left[i(\Delta z - b_1 \cdot \nabla z - b_2 z - b_3 Z) \right] + |\nabla Z|^2 \right\} dx d\sigma \\
&\leq 2\mathbb{E} \int_t^s \left\{ (|b_1|_{W^{1,\infty}(G;\mathbb{R}^n)}^2 + |b_3|_{W^{1,\infty}(G)}^2 + 1) |\nabla z|_{L^2(G)}^2 \right. \\
&\quad \left. + (|b_2|_{W^{1,\infty}(G)}^2 + |b_3|_{W^{1,\infty}(G)}^2 + 1) |z|_{L^2(G)}^2 + |Z|_{H_0^1(G)}^2 \right\} d\sigma.
\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we get

$$\begin{aligned}
& \mathbb{E}|z(t)|_{H_0^1(G)}^2 - \mathbb{E}|z(s)|_{H_0^1(G)}^2 \\
&\leq 2(r_1 + 1) \mathbb{E} \int_t^s |z(\sigma)|_{H_0^1(G)}^2 d\sigma + \mathbb{E} \int_t^s |Z(\sigma)|_{H_0^1(G)}^2 d\sigma.
\end{aligned} \tag{2.10}$$

From (2.10), and thanks to Gronwall's inequality, we arrive at

$$\mathbb{E}|y(t)|_{H_0^1(G)}^2 \leq e^{2(r_1+1)} \left\{ \mathbb{E}|y(s)|_{H_0^1(G)}^2 + \mathbb{E} \int_0^\tau |Z|_{H_0^1(G)}^2 d\sigma \right\}, \tag{2.11}$$

which implies the inequality (2.7) immediately. \square

Remark 2.1 *The proof of this proposition is almost standard. Indeed, if we regard z as a solution to a forward stochastic Schrödinger equation with a nonhomogeneous term Z , then it is a standard energy estimate for such kind of equation.*

Next, we give a proof of Proposition 1.1. For this, we first recall a pointwise identity. For simplicity, in what follows, we adopt the notation $z_i \equiv z_i(x) \triangleq \frac{\partial z(x)}{\partial x_i}$, where x_i is the i -th coordinate of a generic point $x = (x_1, \dots, x_n)$ in \mathbb{R}^n . In a similar manner, we use the notation y_i, v_i , etc., for the partial derivatives of y and v with respect to x_i . Let us recall the following identity.

Lemma 2.1 [18, Proposition 2.3] *Let $\mu = \mu(x) = (\mu^1, \dots, \mu^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field of class C^1 and z an $H_{loc}^2(\mathbb{R}^n)$ -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process. Then for a.e. $x \in \mathbb{R}^n$ and P -a.s. $\omega \in \Omega$, it holds that*

$$\begin{aligned}
& \mu \cdot \nabla \bar{z}(idz + \Delta z dt) + \mu \cdot \nabla z(-id\bar{z} + \Delta \bar{z} dt) \\
&= \nabla \cdot \left[(\mu \cdot \nabla \bar{z}) \nabla z + (\mu \cdot \nabla z) \nabla \bar{z} - i(z d\bar{z}) \mu - |\nabla z|^2 \mu \right] dt + d(i\mu \cdot \nabla \bar{z} z) \\
&\quad - 2 \sum_{j,k=1}^n \mu_j^k z_j \bar{z}_k dt + (\nabla \cdot \mu) |\nabla z|^2 dt + i(\nabla \cdot \mu) z d\bar{z} - i(\mu \cdot \nabla d\bar{z}) dz.
\end{aligned} \tag{2.12}$$

By virtue of Lemma 2.1, the proof of Proposition 1.1 is standard. We only give a sketch here.

Sketch of the Proof of Proposition 1.1 : Since Γ is C^2 , one can find a vector field $\mu_0 = (\mu_0^1, \dots, \mu_0^n) \in C^1(\overline{G}; \mathbb{R}^n)$ such that $\mu_0 = \nu$ on Γ (see [10, page 18] for the construction of μ_0). Letting $\mu = \mu_0$ and $z = y$ in Lemma 2.1, integrating it in Q and taking the expectation, by means of Proposition 2.1, with similar computation in [23], Proposition 1.1 can be obtained immediately. \square

Next, we recall an identity in the spirit of (2.12) but much more complex, which will play an important role in establishing the Carleman estimate for (1.4).

Let $\beta(t, x) \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$, and let $b^{jk}(t, x) \in C^{1,2}(\mathbb{R}^{1+n}; \mathbb{R})$ satisfy

$$b^{jk} = b^{kj}, \quad j, k = 1, 2, \dots, n. \quad (2.13)$$

Let us define a (formal) second order stochastic partial differential operator \mathcal{P} as

$$\mathcal{P}z \triangleq i\beta(t, x)dw + \sum_{j,k=1}^n (b^{jk}(t, x)w_j)_k dt, \quad i = \sqrt{-1}. \quad (2.14)$$

We have the following equality concerning \mathcal{P} :

Lemma 2.2 [18, Theorem 3.1] *Let $\ell, \Psi \in C^2(\mathbb{R}^{1+n}; \mathbb{R})$ and $\theta = e^\ell$. Assume that w is an $H_{loc}^2(\mathbb{R}^n, \mathbb{C})$ -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process. Put $v = \theta w$. Then for a.e. $x \in \mathbb{R}^n$ and P -a.s. $\omega \in \Omega$, it holds that*

$$\begin{aligned} & \theta(\mathcal{P}w\overline{I_1} + \overline{\mathcal{P}w}I_1) + dM + \operatorname{div} V \\ &= 2|I_1|^2 dt + \sum_{j,k=1}^n c^{jk}(v_k \bar{v}_j + \bar{v}_k v_j) dt + D|v|^2 dt \\ &+ i \sum_{j,k=1}^n \left[(\beta b^{jk} \ell_j)_t + b^{jk}(\beta \ell_t)_j \right] (\bar{v}_k v - v_k \bar{v}) dt \\ &+ i \left[\beta \Psi + \sum_{j,k=1}^n (\beta b^{jk} \ell_j)_k \right] (\bar{v} dv - v d\bar{v}) \\ &+ (\beta^2 \ell_t) dv d\bar{v} + i \sum_{j,k=1}^n \beta b^{jk} \ell_j (dv d\bar{v}_k - dv_k d\bar{v}), \end{aligned} \quad (2.15)$$

where

$$\begin{cases} I_1 \triangleq -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v, \\ A \triangleq \sum_{j,k=1}^n b^{jk} \ell_j \ell_k - \sum_{j,k=1}^n (b^{jk} \ell_j)_k - \Psi, \end{cases} \quad (2.16)$$

$$\left\{ \begin{array}{l} M \triangleq \beta^2 \ell_t |v|^2 + i\beta \sum_{j,k=1}^n b^{jk} \ell_j (\bar{v}_k v - v_k \bar{v}), \\ V \triangleq [V^1, \dots, V^k, \dots, V^n], \\ V^k \triangleq -i\beta \sum_{j=1}^n \left[b^{jk} \ell_j (v d\bar{v} - \bar{v} dv) + b^{jk} \ell_t (v_j \bar{v} - \bar{v}_j v) dt \right] \\ \quad - \Psi \sum_{j=1}^n b^{jk} (v_j \bar{v} + \bar{v}_j v) dt + \sum_{j=1}^n b^{jk} (2A \ell_j + \Psi_j) |v|^2 dt \\ \quad + \sum_{j,j',k'=1}^n \left(2b^{jk'} b^{j'k} - b^{jk} b^{j'k'} \right) \ell_j (v_{j'} \bar{v}_{k'} + \bar{v}_{j'} v_{k'}) dt, \end{array} \right. \quad (2.17)$$

and

$$\left\{ \begin{array}{l} c^{jk} \triangleq \sum_{j',k'=1}^n \left[2(b^{j'k} \ell_{j'})_{k'} b^{jk'} - (b^{jk} b^{j'k'} \ell_{j'})_{k'} \right] - b^{jk} \Psi, \\ D \triangleq (\beta^2 \ell_t)_t + \sum_{j,k=1}^n (b^{jk} \Psi_k)_j + 2 \left[\sum_{j,k=1}^n (b^{jk} \ell_j A)_k + A \Psi \right]. \end{array} \right. \quad (2.18)$$

3 A global Carleman estimate for the equation (1.4)

In this section, we establish a global Carleman estimate for the solution to (1.4) (see Theorem 3.1 below).

To begin with, let us introduce the weight functions to be used in our Carleman estimate. Let

$$\psi(x) = |x - x_0|^2 + \sigma, \quad (3.1)$$

where σ is a positive constant such that $\psi \geq \frac{5}{6} |\psi|_{L^\infty(G)}$. Let $s > 0$ and $\lambda > 0$. Put

$$\ell = s \frac{e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}}{t^2(T-t)^2}, \quad \varphi = \frac{e^{4\lambda\psi}}{t^2(T-t)^2}. \quad (3.2)$$

We have the following global Carleman inequality.

Theorem 3.1 *According to (3.2), there is an $s_1 > 0$ (depending on r_1) and a $\lambda_1 > 0$ such that for each $s \geq s_1$, $\lambda \geq \lambda_1$ and for any solution of the equation (1.4), it holds that*

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \left(s^3 \lambda^4 \varphi^3 |z|^2 + s \lambda \varphi |\nabla z|^2 \right) dx dt \\ & \leq C \left[\mathbb{E} \int_Q \theta^2 \left(s^2 \lambda^2 \varphi^2 |Z|^2 + |\nabla Z|^2 \right) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \right]. \end{aligned} \quad (3.3)$$

Proof of Theorem 3.1: The proof is divided into three steps.

Step 1. We choose $\beta = 1$ and $(b^{jk})_{1 \leq j, k \leq n}$ to be the identity matrix. Put

$$\delta^{jk} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Applying Lemma 2.2 to the equation (1.4) with θ given by (3.2), w replaced by z and $v = \theta z$, we obtain that

$$\begin{aligned} & \theta \mathcal{P}z(i\beta \ell_t \bar{v} - 2 \sum_{j,k=1}^n b^{jk} \ell_j \bar{v}_k + \Psi \bar{v}) + \theta \overline{\mathcal{P}z}(-i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v) \\ & + dM + \operatorname{div} V \\ = & 2 \left| -i\beta \ell_t v - 2 \sum_{j,k=1}^n b^{jk} \ell_j v_k + \Psi v \right|^2 dt + \sum_{j,k=1}^n c^{jk} (v_k \bar{v}_j + \bar{v}_k v_j) dt + D |v|^2 dt \\ & + 2i \sum_{j=1}^n (\ell_{jt} + \ell_{tj}) (\bar{v}_j v - v_j \bar{v}) dt + i(\Psi + \Delta \ell) (\bar{v} dv - v d\bar{v}) \\ & + \ell_t dv d\bar{v} + i \sum_{j=1}^n \ell_j (d\bar{v}_j dv - dv_j d\bar{v}). \end{aligned} \tag{3.4}$$

Here

$$\begin{aligned} M &= \beta^2 \ell_t |v|^2 + i\beta \sum_{j,k=1}^n b^{jk} \ell_j (\bar{v}_k v - v_k \bar{v}) \\ &= \ell_t |v|^2 + i \sum_{j=1}^n \ell_j (\bar{v}_j v - v_j \bar{v}); \end{aligned} \tag{3.5}$$

$$\begin{aligned} A &= \sum_{j,k=1}^n b^{jk} \ell_j \ell_k - \sum_{j,k=1}^n (b^{jk} \ell_j)_k - \Psi \\ &= \sum_{j=1}^n (\ell_j^2 - \ell_{jj}) - \Psi; \end{aligned} \tag{3.6}$$

$$\begin{aligned} D &= (\beta^2 \ell_t)_t + \sum_{j,k=1}^n (b^{jk} \Psi_k)_j + 2 \left[\sum_{j,k=1}^n (b^{jk} \ell_j A)_k + A \Psi \right] \\ &= \ell_{tt} + \sum_{j=1}^n \Psi_{jj} + 2 \sum_{j=1}^n (\ell_j A)_j + 2A \Psi; \end{aligned} \tag{3.7}$$

$$\begin{aligned} c^{jk} &= \sum_{j',k'=1}^n \left[2(b^{j'k} \ell_{j'})_{k'} b^{jk'} - (b^{jk} b^{j'k'} \ell_{j'})_{k'} \Psi \right] - b^{jk} \\ &= \left[2(b^{kk} \ell_k)_j b^{jj} - \sum_{j'=1}^n (b^{jk} b^{j'j'} \ell_{j'})_{j'} - b^{jk} \Psi \right] \\ &= 2\ell_{jk} - \delta^{jk} \Delta \ell - \delta^{jk} \Psi; \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
V_k &= -i\beta \sum_{j=1}^n \left[b^{jk} \ell_j (vd\bar{v} - \bar{v}dv) + b^{jk} \ell_t (v_j \bar{v} - \bar{v}_j v) dt \right] \\
&\quad - \Psi \sum_{j=1}^n b^{jk} (v_j \bar{v} + \bar{v}_j v) dt + \sum_{j=1}^n b^{jk} (2A\ell_j + \Psi_j) |v|^2 dt \\
&\quad + \sum_{j,j',k'=1}^n \left(2b^{jk'} b^{j'k} - b^{jk} b^{j'k'} \right) \ell_j (v_{j'} \bar{v}_{k'} + \bar{v}_{j'} v_{k'}) dt \\
&= -i \left[\ell_k (vd\bar{v} - \bar{v}dv) + \ell_t (v_j \bar{v} - \bar{v}_j v) dt \right] - \Psi (v_k \bar{v} + \bar{v}_k v) dt + (2A\ell_k + \Psi_k) |v|^2 dt \\
&\quad + 2 \sum_{j=1}^n \ell_j (\bar{v}_j v_k + v_j \bar{v}_k) dt - 2 \sum_{j'=1}^n \ell_k (v_j \bar{v}_{j'}) dt.
\end{aligned} \tag{3.9}$$

Step 2. In this step, we estimate the terms in the right-hand side of the equality (3.4) one by one.

First, from the definition of ℓ , φ (see (3.2)) and the choice of ψ (see (3.1)), we have

$$\begin{aligned}
|\ell_t| &= \left| s \frac{2(2t-T)}{t^3(T-t)^3} (e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}) \right| \\
&\leq \left| s \frac{2(2t-T)}{t^3(T-t)^3} e^{5\lambda|\psi|_{L^\infty(G)}} \right| \\
&\leq \left| s \frac{C}{t^3(T-t)^3} e^{5\lambda\psi} \right| \\
&\leq Cs\varphi^{1+\frac{1}{2}}
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
|\ell_{tt}| &= \left| s \frac{20t^2 - 20tT + 6T^2}{t^4(T-t)^4} (e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^\infty(G)}}) \right| \\
&\leq \left| s \frac{C}{t^4(T-t)^4} e^{5\lambda|\psi|_{L^\infty(G)}} \right| \\
&\leq \left| s \frac{C}{t^4(T-t)^4} e^{8\lambda\psi} \right| \\
&\leq Cs\varphi^2 \leq Cs\varphi^3.
\end{aligned} \tag{3.11}$$

We choose below $\Psi = -\Delta\ell$, then we have that

$$A = \sum_{j=1}^n \ell_j^2 = \sum_{j=1}^n (4s\lambda\varphi\psi_j)^2 = 16s^2\lambda^2\varphi^2|\nabla\psi|^2. \tag{3.12}$$

Hence, we find

$$\begin{aligned}
D &= \ell_{tt} + \sum_{j=1}^n \Psi_{jj} + 2 \sum_{j=1}^n (\ell_j A)_j + 2A\Psi \\
&= \ell_{tt} + \Delta(\Delta\ell) + 2 \sum_{j=1}^n (4s\lambda\varphi\psi_j 16s^2\lambda^2\varphi^2|\nabla\psi|^2)_j - 32s^2\lambda^2\varphi^2|\nabla\psi|^2\Delta\ell \\
&= 384s^3\lambda^4\varphi^3|\nabla\psi|^4 - \lambda^4\varphi O(s) - s^3\varphi^3 O(\lambda^3) + \ell_{tt}.
\end{aligned} \tag{3.13}$$

Recalling that $x_0 \in (\mathbb{R}^n \setminus \overline{G})$, we know that

$$|\nabla\psi| > 0 \quad \text{in } \overline{G}.$$

From (3.13) and (3.11), we know that there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, one can find a constant $s_0 = s_0(\lambda_0)$ so that for any $s > s_0$, it holds that

$$D|v|^2 \geq s^3\lambda^4\varphi^3|\nabla\psi|^4|v|^2. \tag{3.14}$$

Since

$$\begin{aligned}
c^{jk} &= 2\ell_{jk} - \delta^{jk}\Delta\ell - \delta^{jk}\Psi \\
&= 32s\lambda^2\varphi\psi_j\psi_k + 16s\lambda\varphi\psi_{jk},
\end{aligned}$$

we see that

$$\begin{aligned}
&\sum_{j,k=1}^n c^{jk}(v_j\bar{v}_k + v_k\bar{v}_j) \\
&= 32s\lambda^2\varphi \sum_{j,k=1}^n \psi_j\psi_k(v_j\bar{v}_k + v_k\bar{v}_j) + 16s\lambda\varphi \sum_{j,k=1}^n \psi_{jk}(v_j\bar{v}_k + v_k\bar{v}_j) \\
&= 32s\lambda^2\varphi \left[\sum_{j=1}^n (\psi_j v_j) \sum_{k=1}^n (\psi_k \bar{v}_k) + \sum_{k=1}^n (\psi_k v_k) \sum_{j=1}^n (\psi_j \bar{v}_j) \right] + 32s\lambda\varphi \sum_{j=1}^n (v_j \bar{v}_j + \bar{v}_j v_j) \\
&= 64s\lambda^2\varphi |\nabla\psi \cdot \nabla v|^2 + 64s\lambda\varphi |\nabla v|^2 \\
&\geq 64s\lambda\varphi |\nabla v|^2.
\end{aligned} \tag{3.15}$$

Now we estimate the other terms in the right-hand side of the equality (3.4). The first one reads

$$\begin{aligned}
2i \sum_{j=1}^n (\ell_{jt} + \ell_{tj})(\bar{v}_j v - v_j \bar{v}) &= 4i \sum_{j=1}^n s\lambda\psi_j \ell_t (\bar{v}_j v - v_j \bar{v}) \\
&\leq 2s\varphi |\nabla v|^2 + 2s\lambda^2\varphi^3 |\nabla\psi|^2 |v|^2.
\end{aligned} \tag{3.16}$$

The second one satisfies

$$i(\Psi + \Delta\ell)(\bar{v}dv - v d\bar{v}) = 0. \tag{3.17}$$

For estimating the third and the fourth one, we need to take mean value and get that

$$\begin{aligned}\mathbb{E}(\ell_t dv d\bar{v}) &= \mathbb{E}[\ell_t(\theta \ell_t z dt + \theta dz)(\overline{\theta \ell_t z dt + \theta dz})] = \mathbb{E}(\ell_t \theta^2 dz d\bar{z}) \\ &\leq 2s\theta^2 \varphi^{\frac{3}{2}} \mathbb{E}|Z|^2 dt.\end{aligned}\tag{3.18}$$

Here we utilize the inequality (3.10).

Further,

$$\begin{aligned}\mathbb{E}(d\bar{v}_j dv) &= \mathbb{E}[\overline{(\theta \ell_t z dt + \theta dz)}_j (\theta \ell_t z dt + \theta dz)] \\ &= \mathbb{E}[\overline{(\theta dz)}_j (\theta dz)] \\ &= \mathbb{E}[\overline{(s\lambda \varphi \psi_j \theta dz + \theta dz_j)} \theta dz] \\ &= s\lambda \varphi \psi_j \theta^2 \mathbb{E} d\bar{z} dz + \theta^2 \mathbb{E} d\bar{z}_j dz \\ &= s\lambda \varphi \psi_j \theta^2 \mathbb{E}|Z|^2 dt + \theta^2 \mathbb{E}(\bar{Z}_j Z) dt.\end{aligned}$$

Similarly, we can get that

$$\mathbb{E}(dv_j d\bar{v}) = s\lambda \varphi \psi_j \theta^2 \mathbb{E}|Z|^2 dt + \theta^2 \mathbb{E}(\bar{Z} Z_j) dt.$$

Therefore, the fourth one satisfies that

$$i\mathbb{E} \sum_{j=1}^n \ell_j (d\bar{v}_j dv - dv_j d\bar{v}) = is\lambda \varphi \psi_j [\theta^2 \mathbb{E}(\bar{Z}_j Z) dt - \theta^2 \mathbb{E}(\bar{Z} Z_j) dt].\tag{3.19}$$

Step 3. Integrating the equality (3.4) in Q , taking mean value in both sides, and noting (3.12)–(3.19), we obtain that

$$\begin{aligned}&\mathbb{E} \int_Q \left(s^3 \lambda^4 \varphi^3 |v|^2 + s\lambda^2 \varphi |\nabla v|^2 \right) dx dt + 2\mathbb{E} \int_Q \left| -i\ell_t v - 2 \sum_{j=1}^n \ell_j v_j + \Psi v \right|^2 dx dt \\ &\leq \mathbb{E} \int_Q \left[\theta \mathcal{P} y \left(i\ell_t \bar{v} - 2 \sum_{j=1}^n \ell_j \bar{v}_j + \Psi \bar{v} \right) + \theta \overline{\mathcal{P} y} \left(-i\ell_t v - 2 \sum_{j=1}^n \ell_j v_j + \Psi v \right) \right] dx \\ &\quad + C\mathbb{E} \int_Q \theta^2 (s^2 \lambda^2 \varphi^2 |Z| + |\nabla Z|^2) dx dt + \mathbb{E} \int_Q dM dx + \mathbb{E} \int_Q \operatorname{div} V dx.\end{aligned}\tag{3.20}$$

Now we analyze the terms in the right-hand side of the inequality (3.20) one by one.

The first one reads

$$\begin{aligned}&\mathbb{E} \int_Q \left[\theta \mathcal{P} z \left(i\ell_t \bar{v} - 2 \sum_{j=1}^n \ell_j \bar{v}_j + \Psi \bar{v} \right) + \theta \overline{\mathcal{P} z} \left(-i\ell_t v - 2 \sum_{j=1}^n \ell_j v_j + \Psi v \right) \right] dx \\ &= \mathbb{E} \int_Q \left[\theta (b_1 \cdot \nabla z + b_2 z + b_3 Z) \left(i\ell_t \bar{v} - 2 \sum_{j=1}^n \ell_j \bar{v}_j + \Psi \bar{v} \right) \right. \\ &\quad \left. + \theta (-b_1 \cdot \nabla \bar{z} + b_2 \bar{z} + b_3 \bar{Z}) \left(-i\ell_t v - 2 \sum_{j=1}^n \ell_j v_j + \Psi v \right) \right] dx dt \\ &\leq 2\mathbb{E} \int_Q \left\{ \theta^2 |b_1 \cdot \nabla z + b_2 z + b_3 Z|^2 + \left| -i\beta \ell_t v - 2 \sum_{j=1}^n \ell_j v_j + \Psi v \right|^2 \right\} dx dt.\end{aligned}\tag{3.21}$$

From the choice of θ , we know that $v(0) = v(T) = 0$. Hence, we have

$$\int_Q dM dx = 0. \quad (3.22)$$

Further, by Stokes' Theorem and noting that $v = z = 0$ on $(0, T) \times \Gamma$, we find

$$\begin{aligned} \mathbb{E} \int_Q \operatorname{div} V dx &= \mathbb{E} \int_{\Sigma} 2 \sum_{k=1}^n \sum_{j=1}^n [\ell_j (\bar{v}_j v_k + v_j \bar{v}_k) \nu^k - \ell_k \nu_k v_j \bar{v}_j] d\Sigma \\ &= \mathbb{E} \int_{\Sigma} \left(4 \sum_{j=1}^n \ell_j \nu_j \left| \frac{\partial v}{\partial \nu} \right|^2 - 2 \sum_{k=1}^n \ell_k \nu_k \left| \frac{\partial v}{\partial \nu} \right|^2 \right) d\Sigma \\ &= \mathbb{E} \int_{\Sigma} 2 \sum_{k=1}^n \ell_k \nu_k \left| \frac{\partial v}{\partial \nu} \right|^2 d\Sigma \\ &\leq C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt. \end{aligned} \quad (3.23)$$

From (3.20)–(3.23), we have

$$\begin{aligned} &\mathbb{E} \int_Q \left(s^3 \lambda^4 \varphi^3 |v|^2 + s \lambda \varphi |\nabla v|^2 \right) dx dt \\ &\leq C \mathbb{E} \int_Q \theta^2 |b_1 \cdot \nabla z + b_2 z + b_3 Z|^2 dx dt + C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \\ &\quad + C \mathbb{E} \int_Q \theta^2 (s^2 \lambda^2 \varphi^2 |Z|^2 + |\nabla Z|^2) dx dt. \end{aligned} \quad (3.24)$$

Noting that $z_i = \theta^{-1}(v_i - \ell_i v) = \theta^{-1}(v_i - s \lambda \varphi \psi_i v)$, we get

$$\theta^2 (|\nabla z|^2 + s^2 \lambda^2 \varphi^2 |z|^2) \leq C (|\nabla v|^2 + s^2 \lambda^2 \varphi^2 |v|^2). \quad (3.25)$$

Therefore, it follows from (3.24) that

$$\begin{aligned} &\mathbb{E} \int_Q \left(s^3 \lambda^4 \varphi^3 |z|^2 + s \lambda \varphi |\nabla z|^2 \right) dx dt \\ &\leq C \mathbb{E} \int_Q \theta^2 (|b_1|^2 |\nabla z|^2 + b_2^2 |z|^2 + b_3^2 |Z|^2) dx dt + C \mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 s \lambda \varphi \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \\ &\quad + C \mathbb{E} \int_Q \theta^2 (s^2 \lambda^2 \varphi^2 |Z|^2 + |\nabla Z|^2) dx dt. \end{aligned} \quad (3.26)$$

Taking $\lambda_1 = \lambda_0$ and $s_1 = \max(s_0, Cr_1)$, and utilizing the inequality (3.26), we conclude the desired inequality (3.3). \square

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by virtue of Theorem 3.1.

Proof of Theorem 1.1: By means of the definition of ℓ and θ (see (3.2)), it holds that

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 \left(\varphi^3 |z|^2 + \varphi |\nabla z|^2 \right) dx dt \\ & \geq \min_{x \in \overline{G}} \left(\varphi \left(\frac{T}{2}, x \right) \theta^2 \left(\frac{T}{4}, x \right) \right) \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (|z|^2 + |\nabla z|^2) dx dt, \end{aligned} \quad (4.1)$$

$$\begin{aligned} & \mathbb{E} \int_Q \theta^2 (\varphi^2 |Z|^2 + |\nabla Z|^2) dx dt \\ & \leq \max_{(x,t) \in \overline{Q}} (\varphi^2(t, x) \theta^2(t, x)) \mathbb{E} \int_Q (|Z|^2 + |\nabla Z|^2) dx dt \end{aligned} \quad (4.2)$$

and that

$$\mathbb{E} \int_0^T \int_{\Gamma_0} \theta^2 \varphi \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \leq \max_{(x,t) \in \overline{Q}} (\varphi(t, x) \theta^2(t, x)) \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt. \quad (4.3)$$

From (3.3) and (4.1)–(4.3), we deduce that

$$\begin{aligned} & \mathbb{E} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_G (|z|^2 + |\nabla z|^2) dx dt \\ & \leq Cr_1 \frac{\max_{(x,t) \in \overline{Q}} (\varphi^2(t, x) \theta^2(t, x))}{\min_{x \in \overline{G}} (\varphi(\frac{T}{2}, x) \theta^2(\frac{T}{4}, x))} \\ & \quad \times \left\{ \mathbb{E} \int_Q (|Z|^2 + |\nabla Z|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \right\} \\ & \leq e^{Cr_1} \left\{ \mathbb{E} \int_Q (|Z|^2 + |\nabla Z|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \right\}. \end{aligned} \quad (4.4)$$

Utilizing (4.4) and (2.7), we obtain that

$$\begin{aligned} & \mathbb{E} \int_G (|z_T|^2 + |\nabla z_T|^2) dx \\ & \leq e^{Cr_1} \left\{ \mathbb{E} \int_Q (|Z|^2 + |\nabla Z|^2) dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt \right\}, \end{aligned} \quad (4.5)$$

which concludes Theorem 1.1 immediately. \square

5 Proof of Theorem 1.2–1.3

This section is addressed to proofs of Theorem 1.2-1.3.

Proof of Theorem 1.2: Since the system (1.2) is linear, we only need to show that the attainable set A_T at time T with initial datum $y(0) = 0$ is $L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$, that is, for any $y_1 \in L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$, we can find a pair of control

$$(u, g) \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathcal{F}}(0, T; H^{-1}(G))$$

such that the solution to the system (1.2) with $y(0) = 0$ satisfies that $y(T) = y_1$ in $L^2_{\mathcal{F}_T}(\Omega; H^{-1}(G))$. We achieve this goal by duality argument.

Let

$$b_1 = -a_1, \quad b_2 = -\operatorname{div}(a_1) + a_2, \quad b_3 = -a_3$$

in the equation (1.4). We set

$$\mathcal{X} \triangleq \left\{ \left(\frac{\partial z}{\partial \nu} \Big|_{\Gamma_0}, Z \right) \mid (z, Z) \text{ solves the equation (1.4) with some } z_T \in L^2(\Omega, \mathcal{F}_T, P; H_0^1(G)) \right\}.$$

Clearly, \mathcal{X} is a linear subspace of $L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathcal{F}}(0, T; H_0^1(G))$. Let us define a linear functional \mathcal{L} on \mathcal{X} as follows:

$$\mathcal{L} \left(\frac{\partial z}{\partial \nu} \Big|_{\Gamma_0}, Z \right) = \mathbb{E} \langle y_1, z_T \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E} \int_0^T \int_G z f dx dt.$$

From Theorem 1.1, we see that \mathcal{L} is a bounded linear functional on \mathcal{X} . By means of the Hahn-Banach theorem, \mathcal{L} can be extended to be a bounded linear functional on the space $L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathcal{F}}(0, T; H^{-1}(G))$. For simplicity, we still use \mathcal{L} to denote this extension. Now, by Riesz Representation theorem, we know that there is a pair of random fields

$$(u, g) \in L^2_{\mathcal{F}}(0, T; L^2(\Gamma_0)) \times L^2_{\mathcal{F}}(0, T; H^{-1}(G))$$

such that

$$\begin{aligned} & \mathbb{E} \langle y_1, z_T \rangle_{H^{-1}(G), H_0^1(G)} - \mathbb{E} \int_0^T \int_G z f dx dt \\ &= \mathbb{E} \int_0^T \int_{\Gamma_0} \frac{\partial z}{\partial \nu} u d\Gamma dt + \mathbb{E} \int_0^T \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt. \end{aligned} \tag{5.1}$$

We claim that this pair of random fields (u, g) is the control we need. In fact, from the definition of the solution to (1.2), we have

$$\begin{aligned} & \mathbb{E} \langle y(T), z_T \rangle_{H^{-1}(G), H_0^1(G)} \\ &= \mathbb{E} \int_0^T \int_G z f dx dt + \mathbb{E} \int_0^T \int_{\Gamma_0} \frac{\partial z}{\partial \nu} u d\Gamma dt + \mathbb{E} \int_0^T \langle g, Z \rangle_{H^{-1}(G), H_0^1(G)} dt. \end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we see

$$\mathbb{E}\langle y_1, z_T \rangle_{H^{-1}(G), H_0^1(G)} = \mathbb{E}\langle y(T, \cdot), z_T \rangle_{H^{-1}(G), H_0^1(G)}. \quad (5.3)$$

Since z_T can be arbitrary element in $L_{\mathcal{F}_T}^2(\Omega; H_0^1(G))$, from the equality (5.3), we get $y(T) = y_1$ in $H^{-1}(G)$, P -a.s. \square

At last, we prove Theorem 1.3. In order to present the key idea in the simplest way, we only consider a very special case of the system (1.2), that is, $a_1 = 0$, $a_2 = 0$ and $a_3 = 1$. The argument for the general case is very similar.

Proof of Theorem 1.3: Let us assume that $u \equiv 0$. In this case, the system (1.2) is

$$\begin{cases} idy + \Delta y dt = (y + g)dB(t) & \text{in } (0, T) \times G, \\ y(t, 0) = 0 & \text{on } (0, T) \times \Gamma_0, \\ y(0) = y_0 & \text{in } G. \end{cases} \quad (5.4)$$

Since the system (5.4) is linear, we only need to show that the attainable set A_T at time T for the initial datum $y_0 = 0$ is not $L_{\mathcal{F}_T}^2(\Omega; H^{-1}(G))$. The solution of the system (5.4) is

$$y(T) = S(T)y_0 - i \int_0^T S(T-s)[y(s) + g(s)]dB(s). \quad (5.5)$$

Here $\{S(t)\}_{t \geq 0}$ is the semigroup generated by the following operator

$$\begin{cases} D(A) = H^2(G) \cap H_0^1(G), \\ A\varphi = i\Delta\varphi, \quad \forall \varphi \in D(A). \end{cases}$$

From (5.5), we find $\mathbb{E}(y(T)) = \mathbb{E}(S(T)y_0)$. Thus, if we choose a $y_1 \in L_{\mathcal{F}_T}^2(\Omega; H^{-1}(G))$ such that $\mathbb{E}(y_1) \neq 0$, then y_1 is not in A_T , which completes the proof. \square

6 Further comments and open problems

There are plenty of open problems in the topic of this paper. Some of them are particularly relevant and could need important new ideas and further developments:

- **Null and approximate controllability for stochastic Schrödinger equations**

In this paper, we study the exact controllability for stochastic Schrödinger equations. As immediate consequences, we can obtain the null and approximate controllability for the same system. However, in order to get these two kinds of controllability, we have no reasons to use two controls. By the proof of Theorem 1.3, we know that it is not enough to put one control in the diffusion term to get the null or approximate controllability. On the other hand, suggested by the result in [17], we believe one boundary control in the drift term can guarantee the null and approximate controllability of (1.2). If we want to prove this by following the method in this paper, we will meet some essential

difficulty. For example, to get the null controllability, we should prove the following inequality for the solution to (1.4)

$$\mathbb{E}|z(0)|_{H_0^1(G)}^2 \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt. \quad (6.1)$$

However, if we utilize the method in this paper, we only get

$$\mathbb{E}|z(0)|_{H_0^1(G)}^2 \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 d\Gamma dt + \int_0^T |Z|_{H_0^1(G)}^2 dt.$$

There is an additional term containing Z in the right hand side. This term comes from the fact that, in the Carleman estimate, we regard Z as a nonhomogeneous term rather than part of the solution. Hence, it has to appear in the right hand side of the inequality. Therefore, we believe that one should introduce some new technique, for example, a Carleman estimate in which the fact Z is part of the solution is involved, to get rid of the additional term containing Z . However, we do not know how to achieve this goal now.

- **Exact controllability for stochastic Schrödinger equations with less restrictive condition**

In this paper, we get the exact controllability for (1.2) for Γ_0 given by (1.1). It is well known that a sharp sufficient condition for exact controllability for deterministic Schrödinger equations in analytic domain with time invariant lower order terms is that the triple (G, Γ_0, T) satisfies the Geometric Optic Condition introduced in [2] (see [14] for example). It would be quite interesting and challenging to extend this result to the stochastic setting, but it seems that there are lots of things should be done before solving this problem. For instance, the main idea in [14] is as follows:

Based on a diadic decomposition of the Fourier representation of solutions of the Schrödinger equation, the author shows that, to some extent, one can regard them as superposition of an infinite sequence of solutions of wave equations with velocity of propagation tending to infinity. Then, the fact that the Geometric Optic Condition is satisfied for some finite time T suffices for the exact controllability of the Schrödinger equation to hold for all $T > 0$.

If one follows the above idea, then propagation of singularities for stochastic partial differential equations, at least, for stochastic hyperbolic equations, should be established. However, as far as we know, this topic is completely open.

Further, there are some results showing that, in some situations in which the Geometric Optic Condition is not fulfilled in any time T , one can still obtain exact controllability for the Schrödinger equation. For instance, in [9], it is shown that, when the domain G is a square, for any open non-empty subset G_0 of G , the exact controllability of the Schrödinger equation holds in any time T , in the space $L^2(G)$ and with internal controls in $L^2((0, T) \times G_0)$. How to prove such kind of result is very interesting.

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